and that $\tilde{x}_n$, w_n, w'_{n+1} have variance 1, 1, and 2, respectively,

$$E(\hat{x}^2_{n+1}|Y_n) = (\hat{x}_n + u_n)^2 + 1$$

and, hence, the minimizing u_n in Eq. (B1) satisfies

$$4u_n + 2S_{n+1}(\hat{x}_n + u_n) = 0 \tag{B3}$$

and, hence, from Eq. (B1):

$$S_n\hat{x}^2_n + \Sigma_n = \hat{x}^2_n + 1 + \frac{2S^2_{n+1}\hat{x}^2_n}{(2+S_{n+1})^2} + S_{n+1}\left\{\frac{4\hat{x}^2_n}{(2+S_{n+1})^2} + 1\right\} + \Sigma_{n+1} \tag{B4}$$

This implies that

$$S_n = \frac{2S_{n+1}}{2+S_{n+1}} + 1 \tag{B5a}$$

$$\Sigma_n = \Sigma_{n+1} + (1 + S_{n+1}) \tag{B5b}$$

Backward evaluation of Eqs. (B5) shows that $S_n \to 2$ so that $u_n \to -½\hat{x}_n$, while Σ_n accumulates:

$$\Sigma_n \cong 3(N-n) \quad \text{for} \quad N \gg n$$

Restoring the first power of ϵ and substiuting for y_{n+1} in terms of x_{n+1} and w'_{n+1}, for x_{n+1} in terms of x_n, u_n, and w_n, and replacing x_n by $\hat{x}_n + \tilde{x}_n$, we obtain

$$\begin{aligned}\hat{x}_{n+1} = \hat{x}_n + u_n &+ \left[½ - (\epsilon/8)\pi^{(1)}_n\right](\tilde{x}_n + w_n + w'_{n+1})\\ &+ \epsilon F_1\{\hat{x}^2_n + \hat{x}_n\tilde{x}_n + ½\hat{x}_n(\tilde{x}_n + w_n + w'_{n+1})\\ &+ ½\tilde{x}^2_n + (3/28)(\tilde{x}_n + w_n + w'_{n+1})^2 - (1/7)\}\\ &+ \epsilon H_1\{(\hat{x}_n + u_n)(\tilde{x}_n + w_n) + ½(\tilde{x}_n + w_n)^2\\ &+ (1/14)(\tilde{x}_n + w_n + w'_{n+1})^2 - (3/7)\}\end{aligned} \tag{B6}$$

Recalling that in the steady state, neglecting ϵ^2:

$$E(\tilde{x}^2_n|Y_n) = 1 - \epsilon\pi^{(1)}_n$$

$$E(\tilde{x}_n|Y_n) = 3\epsilon\phi$$

where ϕ is given by Eq. (2), we find

$$\begin{aligned}E(\hat{x}^2_{n+1}|Y_n) &= (\hat{x}_n + u_n)^2 + 1 - (3/4)\epsilon\pi^{(1)}_n\\ &+ \epsilon F_1\left[2(\hat{x}^2_n + 1)(\hat{x}_n + u_n) + 3\hat{x}_n\right] + 2\epsilon H_1(\hat{x}_n + u_n)\end{aligned} \tag{B7}$$

This suggests that Eq. (B2), for $N \gg n$, should be replaced by

$$J_n(\hat{x}_n, \pi^{(1)}_n) = 2\hat{x}^2_n + 3(N-n) + \epsilon(K_n\pi^{(1)}_n + L_n\hat{x}_n + M_n\hat{x}^3_n) \tag{B8}$$

Now, neglecting ϵ, $E(\pi^{(1)}_{n+1}|Y_n)$ is found from (3a) to be given by

$$E(\pi^{(1)}_{n+1}|Y_n) = ¼\pi^{(1)}_n - F_1\hat{x}_n + 2H_1(\hat{x}_n + u_n) \tag{B9}$$

whereas from Eq. (B6):

$$E(\hat{x}_{n+1}|Y_n) = \hat{x}_n + u_n + \mathcal{O}(\epsilon) \tag{B10a}$$

$$E(\hat{x}^3_{n+1}|Y_n) = (\hat{x}_n + u_n)^3 + 3(\hat{x}_n + u_n) + \mathcal{O}(\epsilon) \tag{B10b}$$

Next, $E\{J_{n+1}(\hat{x}_{n+1}, \pi^{(1)}_{n+1}|Y_n\}$ is obtained from Eqs. (B8), (B7), (B9), and (B10), and in the recurrence relation

$$J_n(\hat{x}_n, \pi^{(1)}_n) = \hat{x}^2_n + (1 - \epsilon\pi^{(1)}_n) + \min_{u_n}\left[2u^2_n + E\{J_{n+1}(\hat{x}_{n+1}, \pi^{(1)}_{n+1})|Y_n\}\right] \tag{B11}$$

the minimizing u_n differs from the previous $-½\hat{x}_n$ only by terms of order ϵ. Substitution of $-½\hat{x}_n$ for u_n in Eq. (B11) thus introduces an error only of order ϵ^2, which we neglect. Making this substitution and equating coefficients of $\pi^{(1)}_n$, $\hat{x}^3_n$, and $\hat{x}_n$, we obtain

$$K_n = ¼K_{n+1} - (5/2) \tag{B12a}$$

$$M_n = ⅛M_{n+1} + 2F_1 \tag{B12b}$$

$$L_n = ½L_{n+1} + K_{n+1}(H_1 - F_1) + (3/2)M_{n+1} + 2H_1 + 8F_1 \tag{B12c}$$

Backward iteration of Eqs. (B12) shows that K_n, M_n, and L_n approach the limiting values given in Eq. (5).

Finally, the ϵ correction to u_n is obtainable by equating to zero the partial derivative of Eq. (B12) with respect to u_n, the ϵ terms being evaluated at

$$u_n = -½x_n$$

Thus,

$$4u_n + 4(\hat{x}_n + u_n) + 4\epsilon F_1(\hat{x}^2_n + 1) + 4\epsilon H_1 + 2\epsilon KH_1 + \epsilon L + \epsilon M(¾\hat{x}^2_n + 3) = 0 \tag{B13}$$

The resulting u_n is given by Eq. (6).

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Derivation of the Relative Quaternion Differential Equation

S. Vathsal*

Osmania University, Hyderabad, India 500007

Introduction

RECENTLY the error analysis of a strapdown inertial navigation system using unit quaternions has been presented in the local vertical coordinates.[1] Though the differential equation for the relative quaternion between body-fixed coordinates and local vertical coordinates is given in Eq. (1) of Ref. 1, a derivation from fundamentals is not provided there. For a complete understanding of the significance of this model with reference to the inertial rate of the body-fixed coordinate system C_{xyz} and the inertial rate of the local vertical coordinate system, it is essential to derive the equations from definitions.

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*Principal Scientist and Professor, Research and Training Unit for Navigational Electronics; currently, Scientist "E," Head, System Design Group, Directorate of Systems and Advanced Technology, DRDL Hyderabad, 500258, India.

In this Technical Note, a derivation based on the quaternion composition operation is presented. A numerical example is also presented.

Derivation of the Relative Quaternion Differential Equation

Let the quaternion q represent the orientation of the rigid body with respect to the reference system at time t and $q(t+\Delta t)$ at time $t+\Delta t$. Let $\Delta\Phi$ represent the rotation angle in time Δt. Euler has proven that the transformation from the reference coordinates to the body coordinates can be represented by a single rotation Φ about a fixed line with direction cosine l, m, n. Then, the four parameters become $[l, m, n, \Phi]$. A coordinate transformation matrix between the reference frame and the body frame can also be defined in terms of $[l, m, n, \Phi]^T$. Long ago, the quaternion representation of a rigid body rotation was put forth by Hamilton[2] in terms of a general mathematical formulation. Then with reference to $[l, m, n, \Delta\Phi]^T$, a set of Euler symmetric parameters at $(t+\Delta t)$, namely, q_1', q_2', q_3', q_4', can be defined as follows:

$$q_1' = 1\sin(\Delta\Phi/2)$$
$$q_2' = m\sin(\Delta\Phi/2)$$
$$q_3' = n\sin(\Delta\Phi/2)$$
$$q_4' = \cos(\Delta\Phi/2) \tag{1}$$

These four symmetric parameters are not independent but satisfy the constraint equation

$$q_1'^2 + q_2'^2 + q_3'^2 + q_4'^2 = 1 \tag{2}$$

These four parameters can be regarded as the components of a unit quaternion. The unit quaternion is the relatively simple form for combining the parameters for two individual rotations to give the parameters for the product of two rotations. Then

$$q(t+\Delta t) = \begin{bmatrix} q_4' & q_3' & -q_2' & q_1' \\ -q_3' & q_4' & q_1' & q_2' \\ q_2' & -q_1' & q_4' & q_3' \\ -q_1' & -q_2' & -q_3' & q_4' \end{bmatrix} q \tag{3}$$

Here Δt is assumed to be small and $\Delta\Phi = \omega\Delta t$ where ω is the magnitude of the instantaneous angular velocity of the rigid body. One can use the small angle approximation as follows:

$$\cos(\Delta\Phi/2) \approx 1, \qquad \sin(\Delta\Phi/2) \approx ½\omega\Delta t \tag{4}$$

Then

$$q(t+\Delta t) \approx [1 + ½\Omega\Delta t]q(t) \tag{5}$$

where

$$\Omega = \begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & +\omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix} \tag{6}$$

and subscripts $[x, y, z]$ denote the body axes. Then

$$\frac{dq}{dt} = \mathop{\mathcal{L}t}_{\Delta t\to 0} [q(t+\Delta t) - q(t)]/\Delta t = \frac{1}{2}\Omega q \tag{7}$$

Using the notation of composition $\otimes$ of the unit quaternion, Eq. (7) can be rewritten as

$$\frac{dq}{dt} = \frac{1}{2}\omega\otimes q \tag{8}$$

For example, for unit quaternion q and its inverse q^{-1}, one may write

$$q\otimes q^{-1} = \left[\begin{array}{ccc|c} q_4 & q_3 & -q_2 & q_1 \\ -q_3 & q_4 & q_1 & q_2 \\ q_2 & -q_1 & q_4 & q_3 \\ \hline -q_1 & -q_2 & -q_3 & q_4 \end{array}\right] \left[\begin{array}{c} -q_1 \\ -q_2 \\ -q_3 \\ \hline q_4 \end{array}\right] = \left[\begin{array}{c} \mathbf{0} \\ \hline 1 \end{array}\right] \tag{9}$$

where

$$q^{-1} = \left[\begin{array}{c} -q_1 \\ -q_2 \\ -q_3 \\ \hline -q_4 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c} \mathbf{0} \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 1 \end{array}\right] \tag{10}$$

Let q_b be the unit quaternion defining the transformation from an inertial coordinate system to a body-fixed system, say C_{xyz}. Let q_n be the unit quaternion defining the transformation from the same inertial system to the navigation coordinate system, which could be a local level coordinate system. Then, the following differential equations for q_b and q_n can be derived from Eq. (3):

$$\dot{q}_b = ½\omega_b\otimes q_b \tag{11}$$

$$\dot{q}_n = ½\omega_n\otimes q_n \tag{12}$$

Let q_{bn} be the relative unit quaternion which defines the rotation from navigation to body-fixed coordinates so that

$$q_{bn} = (q_n)^{-1}\otimes q_b \tag{13}$$

By differentiating Eq. (13), one gets

$$\dot{q}_{bn} = \dot{q}_n^{-1}\otimes q_b + q_n^{-1}\otimes\dot{q}_b \tag{14}$$

Also, from Eqs. (2) and (3), one may obtain the result

$$\dot{q}_n^{-1} = -½\omega_n\otimes q_n^{-1} \tag{15}$$

Substituting Eq. (15) into Eq. (14), one gets, in quaternion form,

$$\dot{q}_{bn} = ½\omega_b\otimes q_{bn} - ½q_{bn}\otimes\omega_n \tag{16}$$

In matrix form, one gets

$$\dot{q}_{bn} = ½[\Omega_b]q_{bn} - ½[\Omega_n^*]q_{bn} \tag{17}$$

The skew symmetric matrices $[\Omega_b]$ and $[\Omega_n^*]$ are given by

$$[\Omega_b] = \left[\begin{array}{ccc|c} 0 & +\omega_{bz} & -\omega_{by} & \omega_{bx} \\ -\omega_{bz} & 0 & +\omega_{bx} & \omega_{by} \\ +\omega_{by} & -\omega_{bx} & 0 & \omega_{bz} \\ \hline -\omega_{bx} & -\omega_{by} & -\omega_{bz} & 0 \end{array}\right] \tag{18}$$

$$[\Omega_n^*] = \left[\begin{array}{ccc|c} 0 & -\omega_{nz} & \omega_{ny} & \omega_{nx} \\ \omega_{nz} & 0 & -\omega_{nx} & \omega_{ny} \\ -\omega_{ny} & \omega_{nx} & 0 & \omega_{nz} \\ \hline -\omega_{nx} & -\omega_{ny} & -\omega_{nz} & 0 \end{array}\right] \tag{19}$$

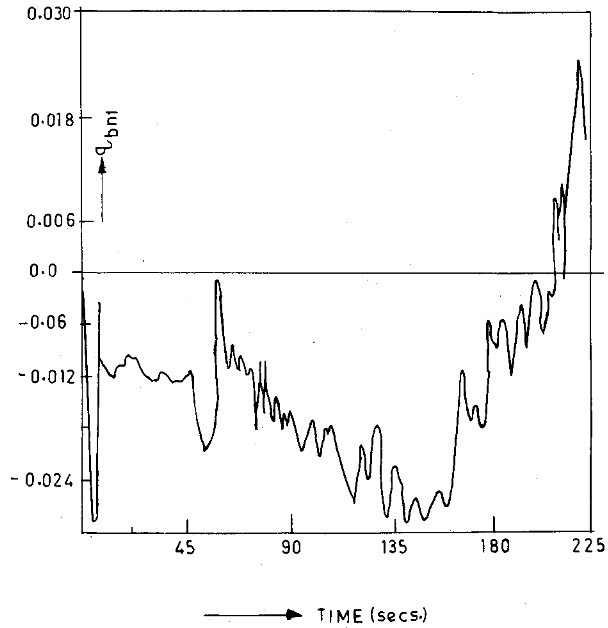

Fig. 1 Relative quaternion q_{bn_1}.

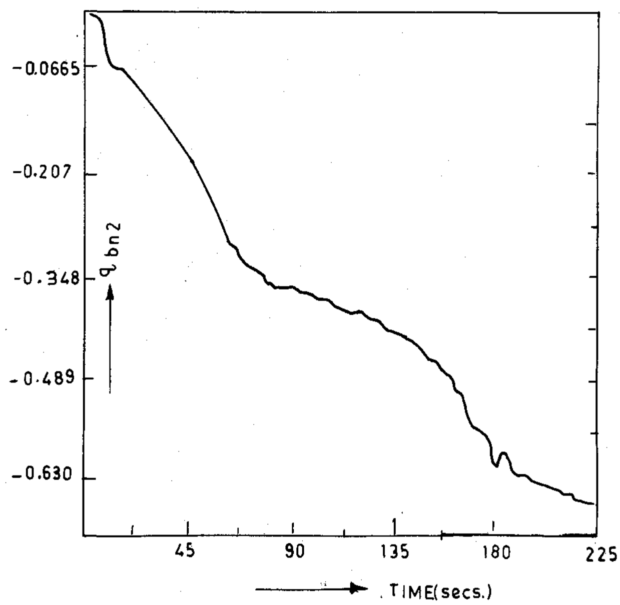

Fig. 2 Relative quaternion q_{bn_2}.

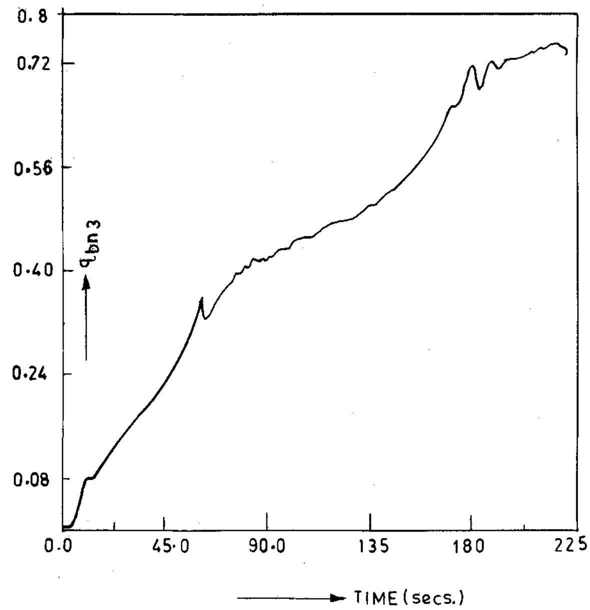

Fig. 3 Relative quaternion q_{bn_3}.

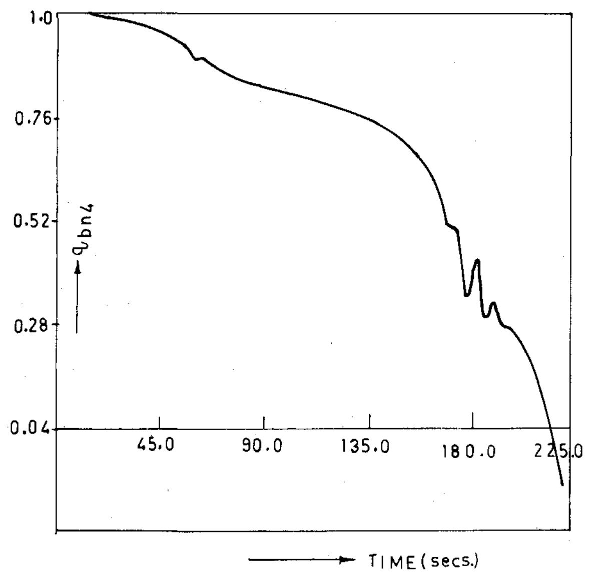

Fig. 4 Relative quaternion q_{bn_4}.

Note that the 3×3 upper left partition of $[\Omega_b]$ and $[\Omega_n^*]$ are different in form. In Ref. 3, $[\Omega_n^*]$ is referred to as the quaternion transmuted matrix. The $[\omega_{bx}, \omega_{by}, \omega_{bz}]$ are to be obtained in the body coordinate system, and $(\omega_{nx}, \omega_{ny}, \omega_{nz})$ are to be obtained in the navigation coordinate system. In Ref. 3, the relative quaternion differential equation has been derived by substituting for ω in Eq. (7) as follows:

$$\omega = \omega_b - C_n^b \omega_n \tag{20}$$

where C_n^b is the coordinate transformation matrix from navigation coordinates to body coordinates. Since elements of C_n^b contain quadratic terms of elements of the unit quaternion, the relative quaternion differential equation becomes nonlinear in q_{bn}. The equation derived in Eq. (17) maintains linearity in q_{bn}, which has advantages in terms of numerical simulation. In the following section, a brief numerical example is given next to illustrate how to find q_{bn}.

Numerical Example

Using Eq. (17), the relative unit quaternion vector q_{bn} has been computed for an experimental strapdown inertial navigation system in which the local vertical coordinates are used for navigation. The body-fixed angular velocity ω_b, obtained from a flight trial, and ω_n are given by

$$\omega_n = \begin{bmatrix} \omega_{nx} \\ \omega_{ny} \\ \omega_{nz} \end{bmatrix} = \begin{bmatrix} \omega_{e_x} + \rho_x \\ \omega_{e_y} + \rho_y \\ \omega_{e_z} + \rho_z \end{bmatrix} = \begin{bmatrix} (|\omega_e| + \dot{\lambda}) \cos\phi \\ -\dot{\phi} \\ -(|\omega_e| + \dot{\lambda}) \sin\phi \end{bmatrix} \tag{21}$$

in the local vertical coordinates; $|\omega_e|$ is the Earth rate; ϕ is the vehicle latitude; and λ is the vehicle longitude. Equation (17) has been integrated using the fourth-order Runge-Kutta scheme of integration in the digital computer. The time variation of elements q_{bn_1} and q_{bn_2} is presented in Figs. 1 and 2, and those of q_{bn_3} and q_{bn_4} are presented in Figs. 3 and 4.

Conclusions

A linear differential equation for the relative quaternion has been derived using the quaternion composition operator. The derivation provides an insight into the rotation matrices of Eq. (17). A numerical example is shown to compute q_{bn} in a digital simulation. This derivation can be extended to other coordinate frames also.

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Existence and Uniqueness Proof for the Minimum Model Error Optimal Estimation Algorithm

D. Joseph Mook*
State University of New York at Buffalo, Buffalo, New York 14260
and
Jiannshiun Lew†
NASA Langley Research Center, Hampton, Virginia 23665

Introduction

RECENTLY, a new approach for performing postexperiment optimal state estimation in the presence of significant model error and/or significant measurement error has been developed.[1] The method has been extended and applied to several examples in postexperiment state estimation, system identification, force estimation, and model error determination.[2-6] The new approach, called minimum model error (MME) estimation, avoids some of the theoretical shortcomings of the commonly used Kalman filter-smoother type of algorithms for postexperiment optimal estimation. These topics are discussed at length in Ref. 1. To obtain the MME estimates, a jump-discontinuous two-point boundary-value problem (TPBVP) must be solved. Although excellent results have been obtained using MME as reported in Refs. 1-6, all of this work simply assumed the existence of a solution and then obtained a solution numerically. The purpose of this Note is to provide an existence and uniqueness proof for the solution of the jump-discontinuous TPBVP of the MME.

Necessary Conditions for Minimum Model Error Optimal Estimator

We define the postexperiment optimal estimation problem as follows: Given a system whose state vector dynamics is modeled by the system of equations,

$$\dot{x} = f[x(t), u(t), t]$$

where $x \equiv n \times 1$ state vector, $f \equiv n \times 1$ vector of dynamic model equations, and $u \equiv p \times 1$ vector of forcing terms, and given a set of discrete measurements modeled by the system of equations,

$$\tilde{z}(\tau_j) = g_j[x(\tau_j), \tau_j] + w_j, \qquad j = 1, \ldots, m$$

where $\tilde{z}(\tau_j) \equiv r \times 1$ measurement set at τ_j, $g_j \equiv r \times 1$ measurement model equations, and $w_j \equiv r \times 1$ measurement error vector, determine the optimal estimate for $x(t)$ during some specified time interval $t_0 \le t \le t_f$. The definition of the optimality criterion is typically the distinguishing feature among various optimal estimation strategies. The optimality criterion for the MME is unique and is developed in Ref. 1, along with a discussion of differences between the MME approach and several other approaches.

In the MME, model error is represented by adding a to-be-determined "unmodeled disturbance," vector $d(t)$ to the right-hand sides of the state model equations as

$$\dot{x} = f[x(t), u(t), t] + d(t)$$

Then, the following cost functional is minimized with respect to $d(t)$:

$$J = \sum_{j=1}^{m} \{[\tilde{z}(\tau_j) - g_j(\hat{x}(\tau_j), \tau_j)]^T R_j^{-1} [\tilde{z}(\tau_j) - g_j(\hat{x}(\tau_j), \tau_j)]\} + \int_{t_0}^{t_f} d(\tau)^T W d(\tau)\, \mathrm{d}\tau$$

where $\hat{x}(\tau_j) \equiv n \times 1$ state vector estimate at τ_j and $W \equiv n \times n$ weight matrix. Determination of the matrix W is discussed in Ref. 1.

The necessary conditions for the minimization of J with respect to $d(t)$ follow directly from a modification[7] of the so-called Pontryagin's necessary conditions[8] and lead to the TPBVP summarized as:

$$\dot{x} = f[x(t), u(t), t] + d(t) \tag{1}$$

$$\dot{\lambda} = -\left(\frac{\partial f}{\partial x}\right)^T \lambda \tag{2}$$

$$d = -\frac{1}{2} W^{-1} \left[\frac{\partial f}{\partial u}\right]^T \lambda \tag{3}$$

$$x(t_0) = \text{specified}, \qquad \text{or} \qquad \lambda(t_0^-) = 0 \tag{4}$$

$$\lambda(\tau_j^+) = \lambda(\tau_j^-) + 2H_j^T R_j^{-1}\{\tilde{z}(\tau_j) - g_j[\hat{x}(\tau_j), \tau_j]\} \tag{5}$$

$$x(t_f) = \text{specified}, \qquad \text{or} \qquad \lambda(t_f^+) = 0 \tag{6}$$

where

$$H \equiv \left.\frac{\partial g_j}{\partial x}\right|_{\hat{x}(\tau_j), \tau_j}$$

The existence of a jump discontinuity in the costate vector λ is evident in Eq. (5) at each measurement time τ_j.

Related Work

The MME algorithm requires solution of the jump-discontinuous TPBVP described by Eqs. (1-6). Despite the large volume of work dealing with the continuous TPBVP,[9,10] relatively little work has been performed on the jump-discontinuous problem. This can be partly explained by available methods for converting "special cases," such as the jump-discontinuous problem, into standard form.[11] However, the solution of the transformed problem is often very inefficient.[12] A general-purpose code (PASVA4)[13] has been developed for nonlinear jump-discontinuous TPBVPs and applied to the problem of seismic ray tracing. However, the code is based on a finite-difference approach, requiring that the mesh points coincide with the jump points. For problems of large order or

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*Assistant Professor, Department of Mechanical and Aerospace Engineering, Clifford C. Furnas Hall. Member AIAA.

†Research Scientist, M.S. 130.